2nd Flow Control Conference
Portland, OR June 28-July 1, 2004

AIAA-2004-2411

Feedback Control for a Two-Dimensional Burgers' Equation System Model

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Abstract

In this paper, we consider the problem of controlling a system governed by a two-dimensional nonlinear partial differential equation. Motivation for the problem is the development of control methodologies for fluid flow, where the dynamics of the system are governed by the nonlinear Navier-Stokes equations. An initial boundary value problem described by the two-dimensional Burgers' equation is formulated to model a right-travelling shock over an obstacle. We focus on implementing feedback control via Dirichlet boundary conditions on the obstacle. We formulate a control problem for the system model, and examine two different methods of finding the control. The first method involves obtaining the solution of an algebraic Riccati equation. The second method involves obtaining a steady-state solution of the Chandrasekhar equations. Numerical approximations are developed to numerically simulate solutions of the problem with and without control. Numerical examples are presented to illustrate the efficacy, as well as the shortcomings, of the control method. Additionally, the influence of boundary condition on the functional gains, and the resulting controls, is demonstrated through numerical examples. Avenues of future work are presented.

Introduction

Closed-loop flow control is a very active area of research. The integration of feedback control with active flow control can enable the stabilization of desired flowfield structures or tracking of flowfield variables. Possible benefits include separation control, virtual aerodynamic shaping, drag reduction, and lift enhancement. Model-based approaches to control law design require a model that captures the dynamics of the relationship between the inputs, the actuators, to the outputs, the variables to be controlled. States that represent additional relationships between the inputs and outputs or between other states are also usually necessary. In addition, the control objective must be quantified in terms of the states. Currently, most model-based approaches to closed-loop flow control consist of either a low-order approach using experimental data or are based on a "reduce then design" philosophy of reducing the order of the model prior to control law design. Both methods can lead to control laws that are either ineffective due to system dynamics that are not captured in the model or are limited in the range of conditions for which they apply. In the reduce/design approach, order reduction techniques are employed to reduce the dimension of a system described by the Navier-Stokes equations, a system of nonlinear partial differential equations, to a system modelled by a much smaller set of ordinary differential equations. The reduced order model is then used to design feedback control laws. This approach has many advantages, foremost of which is the reduction of an infinite-dimensional system to a system consisting of only dozens of states. While potentially powerful, the reduce/design approach has several possible pitfalls with regard to control law design, particularly in the case of boundary control. Due to the order reduction techniques employed, e.g. proper orthogonal decomposition, control laws are developed on a system-by-system basis. Experimental data are taken for a particular system, and a model of the system is developed from those data. As a result, control methods designed for a reduced order model based on a particular configuration and flow condition may not apply to different configurations or flow conditions. Moreover, data used for order reduction, whether from real or simulated experiments, only provide a sampling of the dynamics of a given system. Important dynamics may be lost in the order reduction. Controls deemed effective for the reduced order model may be much less effective when applied to the actual system.

An alternative approach in developing control laws for flow control is based on a "design then reduce" philosophy. In this approach, feedback control laws are developed directly from the governing partial differential equations on the appropriate function spaces. After the optimal feedback control has been determined, order reduction of the system is done. The biggest advantage of the design/reduce approach is that the control law is developed directly from the infinite-dimensional system model. As a result, more of the system dynamics are taken into account before control law design is done, resulting in a more effective control.

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However, implementing controls developed through a design/reduce approach can be computationally expensive. Numerical computations involving feedback control developed in this way for distributed parameter systems typically involve thousands of states, possibly millions in the case of aerodynamic flow control. As a result, implementing these controls is often prohibitively expensive. However, most flow control problems of practical concern involve systems where the number of states greatly outnumbers the number of controls and observations. In systems such as these, computational expense can be greatly reduced by taking advantage of the structure of the problem.

The scalar Burgers' equation, which has a convective nonlinearity like that in the Navier-Stokes momentum equations, is often used for the development of control methods relevant to flow control.[7,8,11] These developments are usually done for the one-dimensional case. Optimal actuator and sensor placement, as well as the control objective, are more complicated issues to consider in the case of higher dimensions. In this paper, we implement a feedback control for a system whose dynamics are governed by a two-dimensional Burgers' equation over a geometry similar to those of interest in flow control problems involving an obstacle. By examining a Burgers' equation system, we illustrate the utility of the control method as well as demonstrate the benefits of utilizing the system structure in the control problem, without having the extra expense associated with flow simulations. Applying these control techniques to an aerodynamic flow control problem is the subject of a future paper.

Problem Description

In this section, we develop a partial differential equation model for the system under consideration. To this end, let $\Omega_1 \subseteq \mathbb{R}^2$ be the open rectangle given by $(a,b) \times (c,d)$. Let $\Omega_2 \subseteq \mathbb{R}^2$ be the rectangle given by $[a_1, a_2] \times [b_1, b_2]$ where $a < a_1 < a_2 < b$ and $c < b_1 < b_2 < d$, i.e., $\Omega_2 \subset \Omega_1$. The problem domain, Ω, is given by $\Omega = \Omega_1 \setminus \Omega_2$. In this configuration, Ω_2 is the obstacle on which we implement Dirichlet boundary control.

The dynamics of the system are given by the two-dimensional Burgers' equation

$$\frac{\partial}{\partial t} w(t,x,y) + K_1 \frac{\partial}{\partial x}\left(\frac{1}{2} w(t,x,y)^2\right) + K_2 \frac{\partial}{\partial y}\left(\frac{1}{2} w(t,x,y)^2\right) = \frac{1}{Re}\left(\frac{\partial^2}{\partial x^2} w(t,x,y) + \frac{\partial^2}{\partial y^2} w(t,x,y)\right) \tag{1}$$

for $t > 0$ and $(x,y) \in \Omega$. Behaviors described by Burgers' equation include shock formation, shock propagation, and rarefaction wave formation. In (1), K_1 and K_2 are constants used to scale the nonlinear terms. The quantity Re is a nonnegative constant, and is analogous to the Reynolds number in the Navier-Stokes equations.

In order to fully specify the model, we need to specify conditions on $\partial\Omega_1$ and $\partial\Omega_2$, as well as an initial condition. As we are interested in implementing control on $\partial\Omega_2$, denote the sides of $\partial\Omega_2$ as

$$\Gamma_1 = \{(x, b_1) \mid a_1 \le x \le a_2\}, \qquad \Gamma_2 = \{(a_1, y) \mid b_1 \le y \le b_2\}, \tag{2}$$

$$\Gamma_3 = \{(x, b_2) \mid a_1 \le x \le a_2\}, \qquad \Gamma_4 = \{(a_2, y) \mid b_1 \le y \le b_2\}.$$

For simplicity, we assume that the controls on $\partial\Omega_2$ are separable, i.e., they are the product of a function of time and a function of the spatial variables. With this assumption, we specify conditions on $\partial\Omega_2$ of the form

$$w(t, \Gamma_1) = \sum_{i=1}^{C_1} u_{1,i}(t)\phi_{1,i}(x), \qquad w(t, \Gamma_2) = \sum_{i=1}^{C_2} u_{2,i}(t)\phi_{2,i}(y), \tag{3}$$

$$w(t, \Gamma_3) = \sum_{i=1}^{C_3} u_{3,i}(t)\phi_{3,i}(x), \qquad w(t, \Gamma_4) = \sum_{i=1}^{C_4} u_{4,i}(t)\phi_{4,i}(y).$$

In (3), C_j is the number of controls on Γ_j. The control $u_{j,i}$ is the i-th control on Γ_j. The function $\phi_{j,i}(\cdot)$ is a function describing the influence of the i-th control on Γ_j.

To complete the model, we specify conditions on $\partial\Omega_1$. Analogous to the no-slip conditions enforced in many flow configurations, we specify that

$$w(t,x,c) = 0 \text{ and } w(t,x,d) = 0. \tag{4}$$

To investigate the influence of the inflow and outflow conditions on the feedback control, we consider the model under four different inflow and outflow conditions. These conditions are of the form

$$w(t,a,y) = f(y) \text{ and } w(t,b,y) = 0, \tag{5}$$

$$w(t,a,y) = f(y) \text{ and } w_x(t,b,y) = 0, \tag{6}$$

$$w(t,a,y) = f(y) \text{ and } 5w(t,b,y) + w_x(t,b,y) = 0, \tag{7}$$

$$w(t,a,y) = w(t,b,y). \tag{8}$$

In (5)-(7), $f(y)$ is a parabolic inlet condition on the left, and is analogous to the inflow condition specified in many channel flow problems. Specifying the value of the solution at the outflow in (5) results in a Dirichlet boundary condition. Specifying the value of the partial derivative of the solution at the outflow in (6) results in a Neumann boundary condition. A Robin's outflow boundary condition is given by (7). Finally, (8) denotes periodic inflow-outflow conditions.

To complete the model, we specify an initial condition of the form

$$w(0,x,y) = w_0(x,y) \in L^2(\Omega). \tag{9}$$

The Control Problem

As our model is described by the two-dimensional Burgers' equation, the system has the potential of being highly nonlinear. There are an extensive number of control techniques available to control linear systems. However, the general theory needed to control nonlinear systems is far from complete. To utilize many of the results from linear control theory, we linearize our PDE system model to obtain an abstract Cauchy problem of the form

$$\frac{\partial}{\partial t} w = Aw + Bu, \ t > 0, \tag{10}$$

$$w(0) = w_0 \in Z, \tag{11}$$

defined on an infinite-dimensional Hilbert space Z. In (10), $A : D(A) \subseteq Z \to Z$ generates a C_0-semigroup, $T(t)$, on Z and B is a (possibly unbounded) linear input operator from the control space U to the state space Z. The controlled output of the system is given by

$$y(t) = Cw(t). \tag{12}$$

We are now in a position to present the control strategy utilized for the system given by (1), (3)-(9). To formulate the control problem, we consider the α-shifted linear quadratic regulator (α-LQR) cost functional

$$J_\alpha(w_0, u) = \int_0^\infty \{\langle Qw, w\rangle_Z + \langle Ru, u\rangle_U\} e^{2\alpha t} dt, \tag{13}$$

where Q is a self-adjoint state weight operator satisfying $Q = C^*C \geq 0$. Similarly, R is a self-adjoint control weight operator satisfying $R > 0$. Finally, the additional performance parameter α satisfies $\alpha \geq 0$. The optimal control problem we consider is to minimize (13) over all controls $u \in L^2((0,\infty); U)$ subject to the constraints (10)-(11). As can be seen, we find the feedback control law from the linearization of the nonlinear problem.

As shown in Refs. 7,8, for an α-controllable system, the α-LQR problem has a unique solution of the form

$$u_{opt} = -Kw \tag{14}$$

$$= -R^{-1}B^*Pw, \tag{15}$$

where P is the unique symmetric, non-negative solution of the algebraic Riccati equation

$$(A+\alpha)^* P + P(A+\alpha) - PBR^{-1}B^*P + Q = 0. \tag{16}$$

It can be shown that the action of the gain operator K in (14) has an integral representation of the form

$$Kw(t,x,y) = \int_\Omega k(x,y)w(t,x,y)dxdy, \tag{17}$$

where the functional gain $k(\cdot,\cdot) \in L^2(\Omega)$.[9] The functional gains indicate how much information each state contributes to the control and provide information about optimal sensor placement. With the functional gains in hand, one can calculate the optimal feedback control by evaluating the integral in (17).

Once the gain operator K is obtained, the linear control law is placed into the nonlinear system according to

$$\frac{\partial}{\partial t} w = (A - BK)w + G(w) + F, \tag{18}$$

$$w(0) = w_0, \tag{19}$$

where $G(w)$ is a nonlinear operator resulting from the convective terms in (1) and F is a forcing term in cases that include the inlet condition $f(y)$.

Discretization Scheme

We utilize a finite difference approximation to numerically solve the initial boundary value problem with and without control. In the work presented here, we specify a uniform grid on Ω with step-size h.

For the second order derivatives in the x and y directions, we implement the standard second order approximations

$$\frac{\partial^2}{\partial x^2} w(x_i, y_j) \simeq \frac{1}{h^2}\left(w_{i+1,j} - 2w_{i,j} + w_{i-1,j}\right), \tag{20}$$

$$\frac{\partial^2}{\partial y^2} w(x_i, y_j) \simeq \frac{1}{h^2}\left(w_{i,j+1} - 2w_{i,j} + w_{i,j-1}\right). \tag{21}$$

Extra care is needed when discretizing the nonlinear terms. At high Re, the convective terms dominate the dynamics of the system and spurious oscillations around the shock occur if an inappropriate numerical scheme is chosen. As discussed in Ref. 14, we utilize a mixture of central differences and a donor cell discretization for the convective terms. These approximations are of the form

$$\begin{aligned} \frac{\partial}{\partial x} w^2(x_i, y_j) \simeq{}& \frac{1}{4h}[(w_{i,j} + w_{i+1,j})^2 - (w_{i-1,j} + w_{i,j})^2] \\ &+ \frac{\gamma}{4h}\left[|w_{i,j} + w_{i+1,j}|\left(w_{i,j} - w_{i+1,j}\right) - |w_{i-1,j} + w_{i,j}|\left(w_{i-1,j} - w_{i,j}\right)\right], \text{ and} \end{aligned} \tag{22}$$

$$\begin{aligned} \frac{\partial}{\partial y} w^2(x_i, y_j) \simeq{}& \frac{1}{4h}[(w_{i,j} + w_{i,j+1})^2 - (w_{i,j-1} + w_{i,j})^2] \\ &+ \frac{\gamma}{4h}\left[|w_{i,j} + w_{i,j+1}|\left(w_{i,j} - w_{i,j+1}\right) - |w_{i,j-1} + w_{i,j}|\left(w_{i,j-1} - w_{i,j}\right)\right], \end{aligned} \tag{23}$$

where $\gamma \in [0, 1]$.

After applying the finite difference approximations given by (20)-(23) and incorporating the conditions on $\partial\Omega_1$ and $\partial\Omega_2$, we obtain a semi-discrete approximation of the form

$$\frac{\partial}{\partial t}w^N = A^N w^N + B^N u + G^N\left(w^N\right) + F^N, \tag{24}$$

$$w^N(0) = w_0^N, \tag{25}$$

where we have utilized superscript N to indicate that we have a finite-dimensional approximation to the infinite-dimensional distributed parameter system.

After implementing the discretization scheme, the finite-dimensional α-LQR problem becomes

$$\min_u \int_0^\infty \left[\left(w^N\right)^T Q\left(w^N\right) + u^T R u\right] e^{2\alpha t} dt \tag{26}$$

$$subject\ to\ \frac{\partial}{\partial t}w^N = A^N w^N + B^N u, \tag{27}$$

$$w^N(0) = w_0^N. \tag{28}$$

In the finite-dimensional case, $Q = C^T C$ is a diagonal, symmetric, positive semi-definite matrix consisting of state weights. R is a diagonal, symmetric, positive definite matrix of control weights.

Solving the finite-dimensional α-LQR problem yields a gain matrix K^N, resulting in an optimal control of the form $u_{opt} = -K^N w^N$. The control is placed into the finite-dimensional system according to

$$\frac{\partial}{\partial t}w^N = \left(A^N - B^N K^N\right) w^N + G^N\left(w^N\right) + F^N, \tag{29}$$

$$w^N(0) = w_0^N. \tag{30}$$

The system (29)-(30) is solved via a 4th order Runge-Kutta method.

Feedback Gain Calculation: Riccati Equation

To numerically simulate the closed-loop system of (29)-(30), one has to calculate the feedback gain matrix K^N. The traditional method of computing K^N involves determining the solution to the algebraic Riccati equation. For this method, the matrix K^N resulting from the finite-dimensional control problem is given by

$$K^N = R^{-1}\left(B^N\right)^T P^N, \tag{31}$$

where P^N is a solution to

$$\left(A^N + \alpha\right)^T P^N + P^N\left(A^N + \alpha\right) - P^N B^N R^{-1}\left(B^N\right)^T P^N + Q^N = 0. \tag{32}$$

Many commercial software packages are available to solve (32). However, numerically solving (32) is a resource intensive undertaking. The symmetric Riccati matrix P^N is typically full, so finding a solution to the nonlinear equation (32) involves finding approximately $\frac{N^2}{2}$ matrix unknowns. For problems where N is very large, as is the case for typical flow control problems, finding a solution to the Riccati equation is prohibitively expensive, if it can be done at all with the computer resources at hand.

Feedback Gain Calculation: Chandrasekhar Equations

For most control problems of practical interest, the number of states is large compared to the number of controls (m) and observations (p). In cases such as these, it is advantageous to find the gain matrix K^N by utilizing the Chandrasekhar equations. These equations are of the form

$$-\dot{K}^N(t) = R^{-1}\left(B^N\right)^T\left(L^N\right)^T(t)L^N(t), \tag{33}$$

$$-\dot{L}^N(t) = L^N(t)\left[\left(A^N+\alpha\right) - B^N K^N(t)\right], \tag{34}$$

with terminal conditions $K(T)=0$ and $L(T)=C$. Equations (33)-(34) are integrated backward in time until a steady-state solution is attained.

Solving the Chandrasekhar equations for the feedback gain matrix K^N is potentially much more efficient as it avoids the expensive calculation of finding P^N by calculating the $N \times m$ unknowns of K^N directly. More detailed discussions of the Chandrasekhar equations can be found in Refs. 6,17. We now present several numerical results.

Example 1

In this example, we illustrate the efficacy of the control strategy. The regions Ω_1 and Ω_2 are specified as $\Omega_1 = (0,1.5)\times(0,.48)$ and $\Omega_2 = [.16,.24]\times[.12,.32]$. The step-size used in the construction of the uniform grid is $h=.02$. The value of γ that we specify in the discretizations (22)-(23) is $\gamma=1$.

We illustrate the system behavior with and without control. For the uncontrolled case, we simply specify that the obstacle boundary, $\partial\Omega_2$, remain fixed at zero as time evolves. In the controlled case, we specify that there are two controls each on the front and back of the obstacle. Similarly, we specify that there is one control each on the bottom and top of the obstacle. For simplicity, we specify that the control influence functions $\{\phi_{j,i}(\cdot)\}$ are piecewise constant.

The control objective is to drive nonzero solution values in a region downstream of the obstacle to zero. To this end, states are weighted throughout the domain with states in the rectangle $[.5,.75]\times[.16,.32]$ being weighted most heavily. The resulting state-weight matrix $Q = C^T C$ has rank one.

Rather than presenting controlled solutions for all four cases of inflow/outflow conditions we are considering, for the sake of brevity we present just the case of parabolic inflow and Neumann outflow, i.e., the condition given by (6). The inlet condition, $f(y)$, is defined as in Fig. 1.

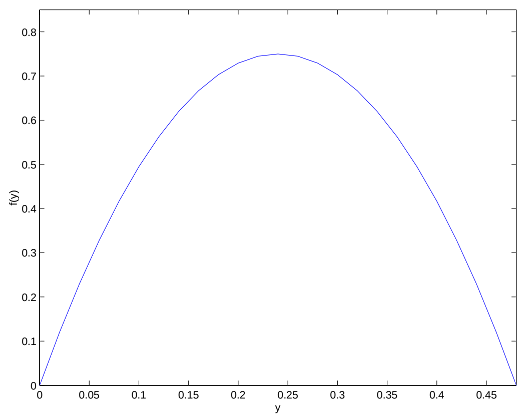

Figure 1: The inlet condition $f(y)$.

The values we specify for K_1 and K_2 in the governing equation are $K_1 = 1$ and $K_2 = 0$. By specifying K_1 and K_2 in this way, the inlet condition will result in solutions which propagate from left to right. In the results presented in this example, the initial condition is defined by $w_0(x,y)\equiv 0$. We set $Re = 500$ and simulate the solution to time $T=10$. The additional performance parameter α in the LQR cost functional is specified to be 0.2. The uncontrolled and controlled solutions are given in Figs. 2-3.

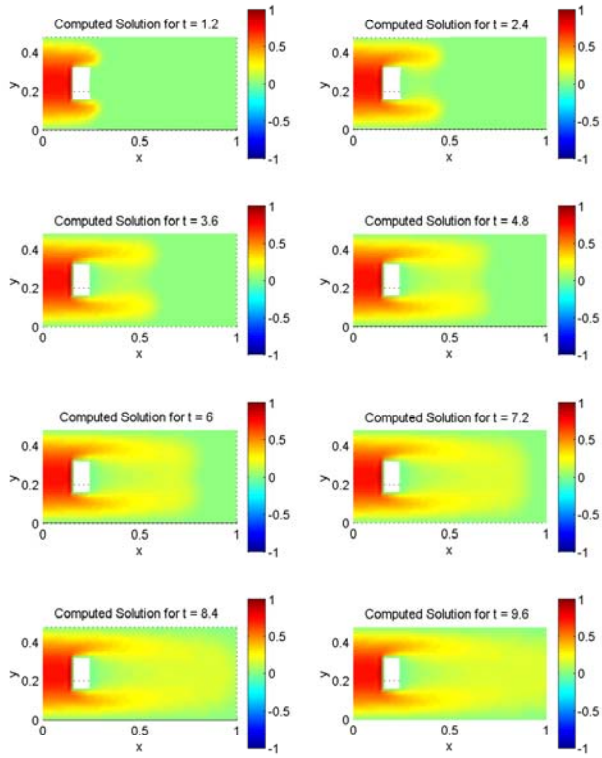

Figure 2: The uncontrolled solution.

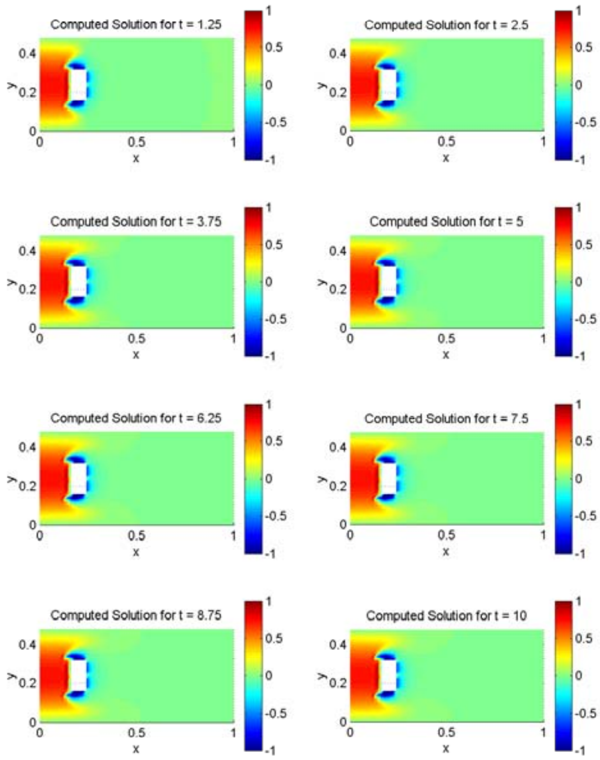

Figure 3: The controlled solution, $\alpha = 0.2$.

As can be seen in Fig. 2, in the uncontrolled case the specification of the inlet condition $f(y)$ results in a solution propagating from left to right. In particular, the solution is a travelling shock. The shock encounters the obstacle, and a portion of it convects above and below. As time evolves, solution values increase downstream of the obstacle until a steady-state solution is reached.

As seen in the controlled case of Fig. 3, the control is very effective at suppressing nonzero states in $[.5, .75] \times [.16, .32]$. A majority of the nonzero states in this region are greatly reduced when compared to their corresponding values in the uncontrolled case. The control objective is satisfied very well.

Example 2

In this example, we investigate the influence of the inflow/outflow conditions specified, as well as the parameter α, on the functional gains of the control problem. Recall that the feedback control can be obtained by evaluating the integral in (17). As is clear from (17), changes in the functional gains have the potential of greatly changing the behavior of the control.

For the case of $\alpha = 0$, there is virtually no difference in the functional gains for the four inflow/outflow conditions, (5)-(8), under consideration. Of course, $\alpha = 0$ corresponds to the traditional LQR control formulation. In this case, the functional gains obtained are large over the rectangle $[.5, .75] \times [.16, .32]$, the region where states are weighted the most heavily. Gains obtained in this case are shown in Fig. 4.

As the performance parameter α is increased, significant changes in the functional gains become apparent. The gains obtained for the four different inflow/outflow conditions are quite different. The gains for the case $\alpha = 0.2$ are shown in Figs. 5-8.

The gains shown in Figs. 5-7 differ significantly. The inflow/outflow conditions (5)-(7) only differ in the type of outflow condition specified. From the plots of the functional gains, the outflow condition has a significant impact on the control as α increases. Clearly, the outflow condition needs to be chosen carefully. The outflow condition is primarily specified as a numerical convenience in order to allow the computed solution to freely exit the computational domain. However, as evident in this example, the condition specified

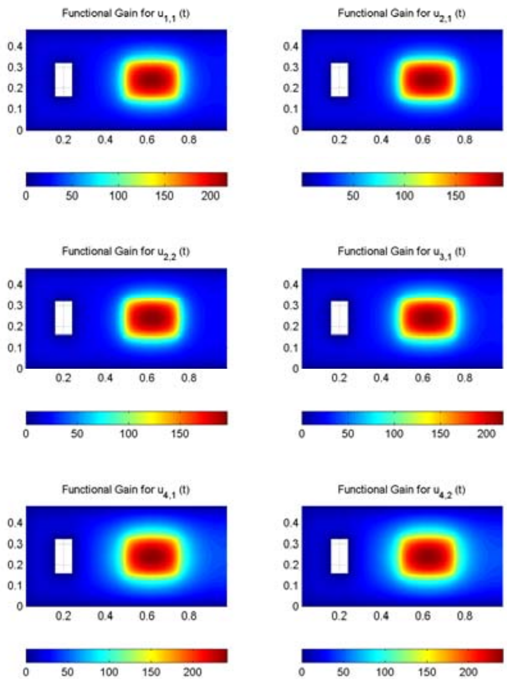

Figure 4: Functional gains obtained for $\alpha = 0$.

for outflow can have a dramatic effect on the functional gains, and hence the resulting controls, depending on the cost functional used in the formulation of the control problem. Careful consideration of the outflow condition must be done in order to avoid controls that are unnecessarily large for the sake of a convenient numerical outflow condition.

Example 3

In this problem, the number of states (N) is not excessively large. As a result, the calculation of the feedback gain matrix K^N can be done by solving the algebraic Riccati equation or by solving the Chandrasekhar equations. The Riccati equation method requires finding approximately $\frac{N^2}{2}$ matrix unknowns. On the other hand, the Chandrasekhar equation method only requires the calculation of $(m+p)N$ unknowns, where m is the number of controls and p is the number of observations. However, the Chandrasekhar equations must be integrated to a steady-state. As the structure of this problem is dominated by the number of states, utilizing the Chandrasekhar equations to find K^N has the potential of providing significant savings. To determine if this is indeed the case, we calculate the gain matrix K^N for the inflow and outflow conditions given by (5)-(8) utilizing both the algebraic Riccati equation as well as the Chandrasekhar equations. Matlab's lqr routine was used to solve the Riccati equation. Similarly, Matlab's ode45 ordinary differential equation solver was used to integrate the Chandrasekhar equations. The results for $\alpha = 0$ are in Table 1.

Table 1: Time comparisons for K^N calculation in CPU minutes, $\alpha = 0$.

Inflow/Outflow Condition	Riccati	Chandrasekhar
(5)	32.09	6.55
(6)	36.05	6.40
(7)	35.52	4.41
(8)	34.35	4.36

As is obvious from Table 1, for the traditional LQR formulation corresponding the case $\alpha = 0$, the Chandrasekhar equation method is able to calculate the gain matrix K^N much more efficiently than was the case for the Riccati equation method. The time required for the calculation is reduced by roughly a factor of 6.

To determine if significant savings are obtainable for $\alpha > 0$, we now specify that $\alpha = 0.2$. In this case, the time required to calculate K^N by the two methods is shown in Table 2.

Table 2: Time comparisons for K^N calculation in CPU minutes, $\alpha = 0.2$.

Inflow/Outflow Condition	Riccati	Chandrasekhar
(5)	35.11	7.38
(6)	38.36	31.75
(7)	34.89	6.36
(8)	33.61	17.15

From Table 2, it is apparent that the Chandrasekhar equation method is still able to calculate the feedback gain matrix more efficiently that the Riccati equation method. However, the savings in the case of $\alpha = 0.2$ are not as great as those obtained in the case of $\alpha = 0$. In particular, the savings obtained for the case of Neumann outflow, namely inflow/outflow condition (6) are greatly reduced. The periodic inflow/outflow condition, i.e., condition (8), also requires significantly more time in the case $\alpha = 0.2$.

It appears that the Chandrasekhar equations are sensitive to the value of α used in the formulation of the control problem, as well as the type of outflow condition specified. To investigate this further, we found K^N by the two methods for the case $\alpha = 0.4$. The results are in Table 3.

Table 3: Time comparisons for K^N calculation in CPU minutes, $\alpha = 0.4$.

Inflow/Outflow Condition	Riccati	Chandrasekhar
(5)	30.52	18.96
(6)	37.40	96.43
(7)	36.29	11.80
(8)	34.65	13.89

From Table 3, it is clear that the Chandrasekhar equation method provides no reduction in computation time in the case of Neumann outflow, i.e. condition (6), when $\alpha = 0.4$. In fact, the Riccati equation method is able to calculate the feedback gain matrix in roughly one-third the time of that required for the Chandrasekhar equations. A much longer time interval is required to integrate the Chandrasekhar equations to a steady-state solution in this particular case. However for the remaining three inflow/outflow conditions, the Chandrasekhar equation method was able to calculate K^N in much less time than that required by the Riccati equation method. As was true in the case of the functional gains, the type of outflow condition has a significant influence on the time required for the gain calculation, depending on the control formulation being used.

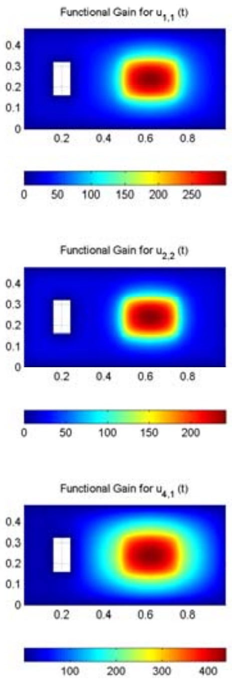

Figure 5: Functional gains for inflow/outflow condition (5), $\alpha = 0.2$.

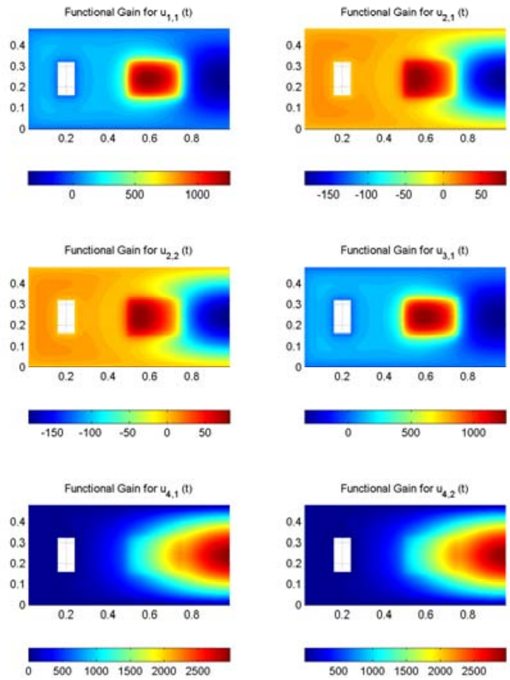

Figure 6: Functional gains for inflow/outflow condition (6), $\alpha = 0.2$.

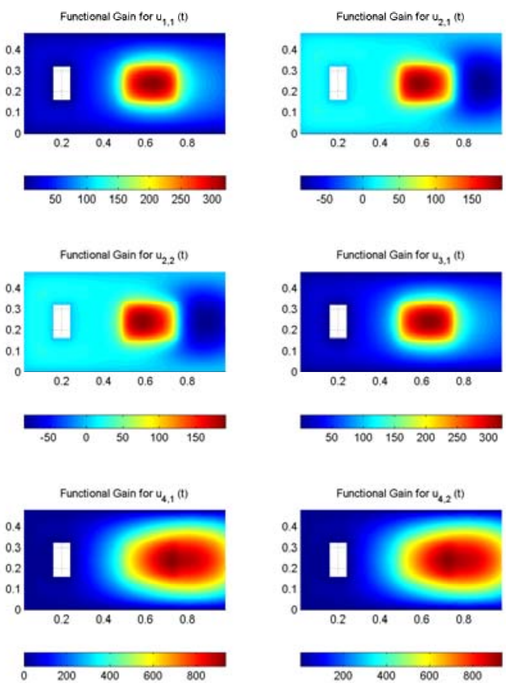

Figure 7: Functional gains for inflow/outflow condition (7), $\alpha = 0.2$.

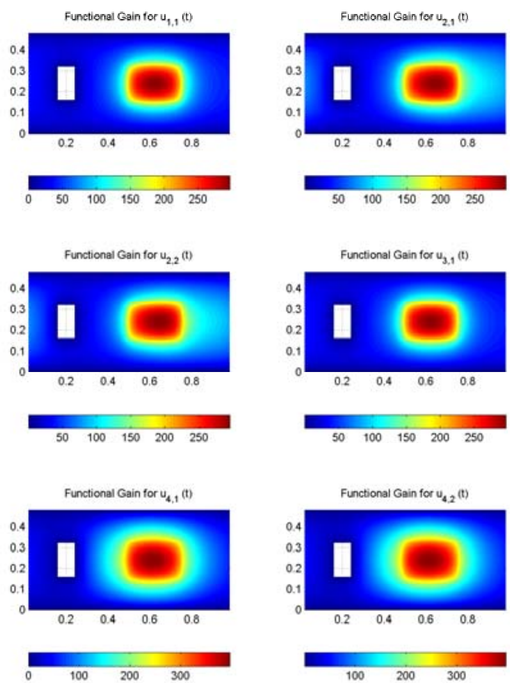

Figure 8: Functional gains for inflow/outflow condition (8), $\alpha = 0.2$.

Conclusions and Future Work

In this paper, we have investigated the effectiveness of a linear quadratic regulator boundary feedback control strategy when applied to a system where the dynamics are governed by a two-dimensional viscous Burgers' equation. We specifically constructed the system in such a way as to incorporate analogous inflow/outflow conditions as those specified in many fluid flow configurations. The strategy chosen to control the system was very effective. As seen in the computational results, the numerical outflow condition can have a dramatic effect on the functional gains and their corresponding controls. The outflow condition needs to be chosen carefully so that unnecessary control effort is not expended solely for the sake of numerical convenience.

We also investigated the benefits of taking system structure into account when calculating optimal feedback controls for systems where the number of states is much greater than the number of controls and observations. Significant reductions in computational time were obtained by solving for the feedback gain matrix directly via the Chandrasekhar equations instead of solving a nonlinear algebraic Riccati equation for most cases we considered. However, the value specified for the additional performance parameter α needs to be chosen carefully to ensure that the resulting control is not excessive and the time required to solve the control problem does not become too great.

As our motivation for considering the problem presented in this paper was an investigation into the utility of linear control techniques to fluid flow problems, future work includes investigating these techniques for the Navier-Stokes equations. There are several issues that must be addressed in that situation. The case of large Reynolds number in the Navier-Stokes equations leads to systems where the dynamics are dominated by nonlinearities. One aspect of future work involves developing nonlinear feedback control methods to overcome this difficulty. In addition, the incompressible Navier-Stokes system leads to a differential algebraic equation (DAE). It is not straightforward to pose a DAE system in a form conducive to control law design. As a result, methods will be developed to formulate the optimal feedback control problem for the DAE system described by the incompressible Navier-Stokes equations.

Acknowledgments

The authors would like to express their appreciation to Dr. Jeff Borggaard of Virginia Polytechnic Institute and State University for his advice and expertise regarding the Chandrasekhar equations.

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